Superfield formulation of central charge anomalies in two-dimensional supersymmetric theories with solitons

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A superfield formulation is presented of the central charge anomaly in quantum corrections to solitons in two-dimensional theories with $N = 1$ supersymmetry. Extensive use is made of the superfield supercurrent, that places the supercurrent J^{μ}_{α}, energy-momentum tensor $\Theta^{\mu\nu}$ and topological current ζ^{μ} in a supermultiplet, to study the structure of supersymmetry and related superconformal symmetry in the presence of solitons. It is shown that the supermultiplet structure of $(J^{\mu}_{\alpha}, \Theta^{\mu\nu}, \zeta^{\mu})$ is kept exact while the topological current ζ^{μ} acquires a quantum modification through the superconformal anomaly. In addition, the one-loop superfield effective action is explicitly constructed to verify the BPS saturation of the soliton spectrum as well as the effect of the anomaly.

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I. INTRODUCTION

There has recently been considerable interest in quantum corrections to solitons in supersymmetric theories in two dimensions. As Witten and Olive [1] pointed out long ago, in supersymmetric theories with topological quantum numbers the supercharge algebra is modified to include central charges and in certain theories the spectrum of topologically stable excitations which saturate the Bogomol'nyi-Prasad-Sommerfield (BPS) bound classically is determined exactly [2]. Solitons (or kinks) in two-dimensional theories with $N = 1$ supersymmetry [3], in particular, are BPS saturated classically and their masses are directly related to the central charges. Whether saturation continues at the quantum level in such theories, however, has been a subject of unforeseen controversy. This is mainly because direct calculations of quantum corrections to the soliton mass and/or central charge, as carried out by a number of authors [4, 5, 6, 7, 8, 9, 10], are a delicate task that requires proper handling of regularization, renormalization and the boundary effect simultaneously; see, e.g., refs. [8, 10] for a survey on earlier research.

It was shown eventually by Shifman, Vainshtein and Voloshin [8] that the central charge is modified by a quantum anomaly so that, together with the quantum correction to the soliton mass, the BPS saturation is maintained at the quantum level. A crucial element in their analysis was the fact that they took advantage of supersymmetry by starting with a superspace action with a supersymmetric regularization prescription, although actual calculations were made with component fields. It is therefore desirable to formulate the problem in a manifestly supersymmetric framework. Recently Fujikawa and van Nieuwenhuizen [10] have developed a superspace approach to the central charge anomaly in two-dimensional theories. They have considered generalized local supersymmetry transformations on the superfield and derived the central charge and related anomalies from the associated Ward-Takahashi identities.

The purpose of this paper is to present a superfield formulation of the central charge and related anomalies in supersymmetric theories from a somewhat different viewpoint. The key point in our approach is to first construct a superfield supercurrent that places the supercurrent J^{μ}_{α}, the energy-momentum tensor $\Theta^{\mu\nu}$ and the topological current ζ^{μ} in a supermultiplet. The superspace Noether theorem for the supercurrent and related superconformal currents is derived from the conservation law of the superfield supercurrent, and possible quantum anomalies are identified to arise from potentially anomalous products that take the form of (fields)×(equations of motion) which, when properly regularized, could yield nonzero results. We take full advantage of supersymmetry by use of the superspace background-field method. It will be shown that the supercharge algebra, or equivalently the supermultiplet structure of $(J^{\mu}_{\alpha}, \Theta^{\mu\nu}, \zeta^{\mu})$, is kept exact while the topological current ζ^{μ} acquires a quantum modification at the one-loop level through the superconformal anomaly. In addition, the one-loop superfield effective action is explicitly constructed to verify the anomaly and BPS saturation of the soliton spectrum.

In Sec. II we review some basic features of supersymmetric theories with topological charges in two dimensions. In Sec. III we construct a superfield supercurrent and related superconformal currents, and examine their conservation laws. In Sec. IV we study the quantum anomalies by use of the superspace background-field method. In Sec. V we calculate the one-loop effective action and examine its consequences. Section VI is devoted to a summary and discussion.

II. N=I SUPERSYMMETRY IN TWO DIMENSIONS

The $N = 1$ superspace consists of points labeled by the space-time coordinates $x^{\mu} = (x^0, x^1)$ and two real Grassmann coordinates $\theta_{\alpha} = (\theta_1, \theta_2)$; we denote $z = (x^{\mu}, \theta_{\alpha})$ collectively. Throughout this paper we adopt the

superspace notation of Ref. [8]. For the Dirac matrices we use the Majorana representation

$$\gamma^0 = \sigma_2, \gamma^1 = i\sigma_3 \tag{2.1}$$

with the charge-conjugation matrix $C_{\alpha\beta} = -(\gamma^0)_{\alpha\beta} = i\epsilon_{\alpha\beta}$ and $\epsilon_{12} = 1$ so that for a charge-conjugate spinor $\psi^C_\alpha \equiv C_{\alpha\beta}\bar{\psi}_\beta = \psi^\dagger_\alpha$ with $\bar{\psi} = \psi^\dagger\gamma^0$; accordingly Majorana spinors such as θ_α are real spinors.

The real superfield $\Phi(z) \equiv \Phi(x,\theta)$ consists of a scalar field $\phi(x)$ and a real spinor field $\psi_\alpha(x)$, along with a real auxiliary field $F(x)$, with the expansion

$$\Phi(x,\theta) = \phi(x) + \bar{\theta}\psi(x) + \frac{1}{2}\,\bar{\theta}\theta\,F(x), \tag{2.2}$$

where $\bar{\theta} \equiv \theta\gamma^0 = i(\theta_2, -\theta_1)$, $\bar{\theta}\theta = \bar{\theta}_\alpha\theta_\alpha = -2i\theta_1\theta_2$ and $\bar{\theta}\psi \equiv \bar{\theta}_\alpha\psi_\alpha = \bar{\psi}_\alpha\theta_\alpha$. Translations in superspace

$$x^\mu \to x^\mu + i\bar{\xi}\gamma^\mu\theta, \quad \theta_\alpha \to \theta_\alpha + \xi_\alpha \tag{2.3}$$

generate the variations of the superfield,

$$\delta\Phi(x,\theta) = \bar{\xi}_\alpha\mathcal{Q}_\alpha\Phi(x,\theta) \tag{2.4}$$

with $\mathcal{Q}_\alpha = \partial/\partial\bar{\theta}_\alpha + i(\gamma^\mu\theta)_\alpha\partial_\mu$. The component fields thereby undergo the supersymmetry transformations

$$\begin{aligned} \delta\phi(x) &= \bar{\xi}\psi(x), \delta\psi_\alpha(x) = -i(\gamma^\mu\xi)_\alpha\partial_\mu\phi(x) + \xi_\alpha F(x), \\ \delta F(x) &= -i\bar{\xi}\gamma^\mu\partial_\mu\psi(x). \end{aligned} \tag{2.5}$$

We consider a supersymmetric model described by a superfield action of the form [3]

$$S = \int d^2x\, d^2\theta\, \left\{\frac{1}{4}\,(\bar{D}_\alpha\Phi)\,D_\alpha\Phi + W(\Phi)\right\} \tag{2.6}$$

with $\int d^2\theta\,\frac{1}{2}\,\bar{\theta}\theta = 1$; the superpotential $W(\Phi)$ is a polynomial in $\Phi(x,\theta)$. Here the spinor derivatives

$$D_\alpha \equiv \frac{\partial}{\partial\bar{\theta}_\alpha} - (\not{p}\,\theta)_\alpha \tag{2.7}$$

with $\not{p} \equiv \gamma^\mu p_\mu$ and $p_\mu = i\partial_\mu$, or their conjugates

$$\bar{D}_\alpha \equiv D_\beta(\gamma^0)_{\beta\alpha} = -\partial/\partial\theta_\alpha + (\bar{\theta}\not{p})_\alpha, \tag{2.8}$$

anticommute with supertranslations, $\{D_\alpha, \mathcal{Q}_\beta\} = 0$, and obey

$$\{D_\alpha, D_\beta\} = 2\,(\not{p}\gamma^0)_{\alpha\beta}; \tag{2.9}$$

see Appendix A for some formulas related to D_α. In component fields the action reads $S = \int d^2x\,\mathcal{L}$ with

$$\mathcal{L} = \frac{1}{2}\,\{\bar{\psi}i\not{\partial}\psi + (\partial_\mu\phi)^2 + F^2\} + FW'(\phi) - \frac{1}{2}\,W''(\phi)\,\bar{\psi}\psi, \tag{2.10}$$

where $W'(\phi) = dW(\phi)/d\phi$, etc. Eliminating the auxiliary field F from $\mathcal{L}$, i.e., setting $F \to -W'(\phi)$, yields the potential term $-\frac{1}{2}\,[W'(\phi)]^2$.

The superpotential $W(\Phi)$ may be left quite general in form. For definiteness we take $W(\Phi)$ to be odd in Φ and to have more than one extrema with $W'(\Phi) = 0$ so that the model supports topologically stable solitons interpolating between different degenerate vacua at spatial infinities $x^1 \to \pm\infty$. A simple choice [3]

$$W(\Phi) = \frac{m^2}{4\lambda}\,\Phi - \frac{\lambda}{3}\,\Phi^3 \tag{2.11}$$

corresponds to the two-dimensional version of the Wess-Zumino model [11], and supports a classical static kink solution

$$\phi^{\rm kink}(x) = (m/2\lambda)\tanh(mx^1/2) \tag{2.12}$$

interpolating between the two vacua with $\langle\phi\rangle_{\rm vac} = \pm m/(2\lambda)$ at $x^1 = \pm\infty$. Similarly, the supersymmetric sine-Gordon model with $W(\Phi) = mv^2\sin(\Phi/v)$ also supports solitons.

Associated with the supertranslations (2.5) of the component fields is the Noether supercurrent

$$J^\mu_\alpha = (\partial_\nu\phi)(\gamma^\nu\gamma^\mu\psi)_\alpha - iF\,(\gamma^\mu\psi)_\alpha. \tag{2.13}$$

The conserved supercharges

$$Q_\alpha = \int dx^1 J^0_\alpha \tag{2.14}$$

generate, within the canonical formalism (and by naive use of equations of motion), the transformation law of the supercurrent [8]

$$i[\bar{\xi}_\beta Q_\beta, J^\mu_\alpha] = 2\,(\gamma_5\xi)_\alpha\zeta^\mu - 2i(\gamma_\lambda\xi)_\alpha\Theta^{\mu\lambda}, \tag{2.15}$$

where

$$\Theta^{\mu\lambda} = \frac{i}{2}\,\bar{\psi}\gamma^\mu\partial^\lambda\psi + \partial^\mu\phi\partial^\lambda\phi - \frac{1}{2}\,g^{\mu\lambda}\{(\partial_\nu\phi)^2 - F^2\} \tag{2.16}$$

is the energy-momentum tensor and

$$\zeta^\mu = -\epsilon^{\mu\nu}F\partial_\nu\phi = \epsilon^{\mu\nu}\partial_\nu W(\phi) \tag{2.17}$$

is the topological current. Here $\gamma_5 \equiv \gamma^0\gamma^1 = -\sigma_1$; note the matrix identities specific to 1+1 dimensions,

$$\gamma^\mu\gamma_5 = -\epsilon^{\mu\nu}\gamma_\nu, \ \gamma^\mu\gamma^\nu = g^{\mu\nu} + \epsilon^{\mu\nu}\gamma_5, \tag{2.18}$$

with $\epsilon^{01} = 1$.

From Eq. (2.15) follows the supersymmetry algebra [1, 12]

$$\{Q_\alpha, \bar{Q}_\beta\} = 2(\gamma_\lambda)_{\alpha\beta}\,P^\lambda + 2i(\gamma_5)_{\alpha\beta}\,Z \tag{2.19}$$

where $P^\lambda = \int dx^1\Theta^{0\lambda}$ is the total energy and momentum and Z is the central charge

$$Z = \int dx^1\zeta^0 = W(\phi)|_{x^1=\infty} - W(\phi)|_{x^1=-\infty}. \tag{2.20}$$

The $N = 1$ superalgebra thus gets centrally extended in the presence of solitons [1].

III. SUPERFIELD SUPERCURRENT

The transformation law (2.15) implies that the currents J^μ_α, ζ^μ and $\Theta^{\mu\lambda}$ form a supermultiplet. It is possible to construct a superfield supercurrent that exposes this multiplet structure. To this end note first that J^μ_α has dimension 3/2 in units of mass while $\Theta^{\mu\lambda}$ and ζ^μ have dimension 2. The supercurrent should thus be a spinor-vector superfield (of dimension 3/2) that contains J^μ_α as the lowest component, in sharp contrast to those in four dimensions [13] that are vector superfields $\sim V_{\alpha\dot\alpha}$. It is quadratic in Φ and contains three D's. The result is

$$\mathcal{V}^\mu_\alpha = -i(D_\alpha \bar D_\lambda \Phi)\,(\gamma^\mu)_{\lambda\beta}\, D_\beta \Phi. \tag{3.1}$$

For formal consideration it is more appropriate to define $\mathcal{V}^\mu_\alpha$ by the equivalent symmetrized form

$$-\frac{i}{2}\left\{ D_\alpha \bar D_\lambda \Phi\cdot(\gamma^\mu)_{\lambda\beta}\, D_\beta\Phi + \bar D_\lambda \Phi\cdot(\gamma^\mu)_{\lambda\beta}\, D_\alpha D_\beta\Phi\right\}, \tag{3.2}$$

but we shall use the notation (3.1) with such symmetrization understood from now on.

The current $\mathcal{V}^\mu_\alpha$ is a real spinor superfield; note, in this connection, that $(D_\alpha)^\dagger = -D_\alpha$ so that $D_\alpha\Phi$, e.g., is a real spinor superfield. In components it reads

$$\mathcal{V}^\mu_\alpha = J^\mu_\alpha - 2i(\gamma_\lambda\theta)_\alpha\Theta^{\mu\lambda} + 2(\gamma_5\theta)_\alpha\zeta^\mu + \theta_\alpha X^\mu + \frac{1}{2}\,\bar\theta\theta f^\mu_\alpha. \tag{3.3}$$

Here $\zeta^\mu = -\epsilon^{\mu\nu}F\partial_\nu\phi$, as before, and

$$X^\mu = -\bar\psi\,\gamma^\mu\gamma^\nu\partial_\nu\psi = i\bar\psi\,\gamma^\mu\frac{\delta S}{\delta\bar\psi}. \tag{3.4}$$

One may be tempted to set $X^\mu = 0$, but some care is needed here. The equation of motion $\delta S/\delta\bar\psi = i\gamma^\nu\partial_\nu\psi - W''(\phi)\psi = 0$ is assumed to hold by itself. In contrast, the equations of motion multiplied by fields are potentially singular and, when properly regulated, may not vanish. In Fujikawa's path-integral formulation of anomalies [14, 15] all known anomalies arise from regularized Jacobian factors which take precisely the form (fields)×(equations of motion). We shall therefore avoid using the equations of motion in our analysis and keep track of the potentially anomalous products such as X^μ; actually, conformal [16] and superconformal [17] anomalies in four dimensions were studied along this line earlier. In the next section we evaluate such anomalous products by regularizing them in a supersymmetric way.

The highest term f^μ_α in $\mathcal{V}^\mu_\alpha$ has a somewhat long expression,

$$\begin{aligned} f^\mu_\alpha &= 2\epsilon^{\mu\lambda}\partial_\lambda(F\gamma_5\psi)_\alpha + r^\mu_\alpha, && (3.5)\\ r^\mu_\alpha &= (\gamma^\lambda\gamma^\mu + [\gamma^\lambda,\gamma^\mu])_{\alpha\beta}\Big(\psi_\beta\partial_\lambda\frac{\delta S}{\delta F} + \frac{\delta S}{\delta\bar\psi_\beta}\,\partial_\lambda\phi\Big) \\ &\quad -i(\gamma^\mu)_{\alpha\beta}\Big(\frac{\delta S}{\delta\bar\psi_\beta}\,F + \psi_\beta\,\frac{\delta S}{\delta\phi}\Big). && (3.6)\end{aligned}$$

The potentially anomalous products X^μ and r^μ_α, vanishing trivially at the classical level, actually continue to vanish at the quantum level, as shown in the next section; anticipating this result (and to avoid inessential complications), we shall set $X^\mu = r^\mu = 0$ in the rest of this section. Correspondingly, $f^0_\alpha = 2\partial_1(F\gamma_5\psi)_\alpha$ is a total divergence so that the associated conserved spinor charge vanishes (so long as the field $\psi(x)\to 0$ as $x^1\to\pm\infty$),

$$\int_{-\infty}^{\infty} dx^1\, f^0_\alpha = 2\Big[F(\gamma_5\psi)_\alpha\Big]_{x^1=-\infty}^{x^1=\infty} = 0. \tag{3.7}$$

Hence only Q_α, P^λ and Z form an irreducible supermultiplet, yielding a conserved-charge superfield

$$\int_{-\infty}^{\infty} dx^1 \mathcal{V}^0_\alpha = Q_\alpha - 2i(\gamma_\lambda\theta)_\alpha P^\lambda + 2(\gamma_5\theta)_\alpha Z. \tag{3.8}$$

Note that, upon supertranslations, Eqs. (3.3) and (3.8) correctly lead to the superalgebras (2.15) and (2.19), respectively.

Let us examine the conservation law of the supercurrent $\mathcal{V}^\mu_\alpha$. Let ∂_μ act on it, rewrite $(\not{p}D)_\alpha$ or $(\bar D\not{p})_\alpha$ in terms of three D's using the formulas in Appendix A, and and eliminate the expression $\bar D D\Phi$ by use of an identity involving the superfield equation of motion

$$\frac{\delta S}{\delta\Phi} = -\frac{1}{2}\,\bar D D\Phi + W'(\Phi). \tag{3.9}$$

The result is

$$\partial_\mu\mathcal{V}^\mu_\alpha = (D_\alpha\bar D_\beta\Phi)\,D_\beta\frac{\delta S}{\delta\Phi} - (D_\alpha\bar D_\beta\frac{\delta S}{\delta\Phi})\,D_\beta\Phi. \tag{3.10}$$

This conservation law is an identity. Accordingly, current conservation $\partial_\mu\mathcal{V}^\mu_\alpha = 0$ follows from the equation of motion $\delta S/\delta\Phi = 0$ at the classical level. Actually $\partial_\mu\mathcal{V}^\mu_\alpha = 0$ also persists at the quantum level, with the potentially anomalous terms vanishing, as we shall see later.

It is enlightening to cast Eq. (3.10) into a somewhat different form. Let us multiply it with an arbitrary real spinor superfield $\Omega_\alpha(z)$ (or $\bar\Omega = \Omega\gamma^0$) and integrate over $z = (x^\mu, \theta_\alpha)$. Making integrations by parts then yields

$$-\int d^4z\,\bar\Omega_\alpha\partial_\mu\mathcal{V}^\mu_\alpha = \int d^4z\,\frac{\delta S}{\delta\Phi}\,\delta_\Omega\Phi, \tag{3.11}$$

where $d^4z \equiv d^2x\,d^2\theta$ and

$$\begin{aligned}\delta_\Omega\Phi &= \bar D_\beta\bar\Omega_\alpha D_\alpha D_\beta\Phi - \bar D_\beta D_\alpha\bar\Omega_\alpha D_\beta\Phi, \\ &= (\bar D_\beta\bar D_\alpha\Omega_\alpha)D_\beta\Phi - 2i(\bar D\gamma^\mu\Omega)\partial_\mu\Phi. \end{aligned}\tag{3.12}$$

This shows that the supercurrent $\mathcal{V}^\mu_\alpha$ is a Noether current associated with the superfield variation $\delta_\Omega\Phi$ and, in addition, that the right-hand side of Eq. (3.10) is essentially a Jacobian factor associated with the field change $\Phi + \delta_\Omega\Phi$ within Fujikawa's method. The $\delta_\Omega\Phi$ generalizes translations in superspace to their local form. Indeed, with the parametrization

$$\Omega_\alpha = \eta_\alpha - \frac{i}{4}(\gamma_\mu\theta)_\alpha a^\mu + \frac{1}{2}\,\theta_\alpha b - \frac{1}{4}(\gamma_5\theta)_\alpha c + \frac{1}{2}\,\bar\theta\theta\,\xi_\alpha, \tag{3.13}$$

one finds on the left-hand side of Eq. (3.11) the conservation laws

$$-\bar{\xi}_\alpha \partial_\mu J^\mu_\alpha - a_\lambda \partial_\nu \Theta^{\nu\lambda} - b\,\partial_\mu X^\mu - c\,\partial_\mu \zeta^\mu - \bar{\eta}_\alpha \partial_\mu f^\mu_\alpha. \quad (3.14)$$

At the same time, the field variation $\delta_\xi \Phi$ caused by $\xi_\alpha(x)$ is seen to generalize supertranslations (2.5) into local ones [with $\delta F = -i\partial_\mu(\bar{\xi}\gamma^\mu\psi)$]. Similarly, $a^\mu(x)$ generates local spacetime translations with the variation

$$\delta_a \Phi = a^\mu \partial_\mu \phi + \bar{\theta}\Big(a^\mu \partial_\mu \psi + \frac{1}{2}\,\psi\,\partial^\mu a_\mu\Big) + \frac{1}{2}\,\bar{\theta}\theta\,\partial_\mu(a^\mu F). \quad (3.15)$$

The topological current ζ^μ is associated with the variation

$$\delta_c \Phi = -\frac{1}{2}\,\bar{\theta}\theta\,\epsilon^{\mu\lambda}(\partial_\mu \phi)\partial_\lambda c \quad (3.16)$$

that affects only F, as noted in ref. [10], while $X^\mu (=0)$ derives from the variation

$$\delta_b \Phi = -i(\bar{\theta}\gamma^\mu\psi)\partial_\mu b \quad (3.17)$$

that affects ψ alone. The field variation $\delta_\eta \Phi$ caused by $\eta_\alpha(x)$ reveals the form of spinorial transformations that give rise to the conserved current f^μ_α; its explicit form, being unilluminating, is suppressed here.

The supercurrent $\mathcal{V}^\mu_\alpha$ is also used to generate superconformal transformations of fields [13]. In particular, the conservation laws of superconformal currents are characterized by the quantity $(\gamma_\mu \mathcal{V}^\mu)_\alpha$, as seen from the simplest example $\partial_\mu(\not{x}\mathcal{V}^\mu)_\alpha = (\gamma_\mu \mathcal{V}^\mu)_\alpha$ for $\partial_\mu \mathcal{V}^\mu_\alpha = 0$. Rewriting $\mathcal{V}^\mu_\alpha$ as

$$\mathcal{V}^\mu_\alpha = (\partial_\nu \Phi)(\gamma^\nu\gamma^\mu)_{\alpha\beta}\, D_\beta \Phi + i\frac{1}{2}(\bar{D}D\Phi)\,(\gamma^\mu)_{\alpha\beta}\, D_\beta \Phi \quad (3.18)$$

and noting $\gamma_\mu\gamma^\lambda\gamma^\mu = 0$ yields $(\gamma_\mu \mathcal{V}^\mu)_\alpha = i(\bar{D}D\Phi)\, D_\alpha \Phi$, which, by use of Eq. (3.9), is cast in the form

$$i(\gamma_\mu \mathcal{V}^\mu)_\alpha + 2 D_\alpha W(\Phi) = 2\,\frac{\delta S}{\delta \Phi}\, D_\alpha \Phi \equiv \mathcal{A}_\alpha. \quad (3.19)$$

This is a supersymmetric version of the trace identity [18], as seen from the component expression

$$\begin{aligned} i(\gamma_\mu \mathcal{V}^\mu)_\alpha \;&=\; i(\gamma_\mu J^\mu)_\alpha + 2\,\theta_\alpha \Theta^\mu_\mu + 2(\gamma_5\theta)_\alpha \epsilon_{\mu\lambda}\Theta^{\mu\lambda} \\ &\quad + 2i(\gamma_\mu\theta)_\alpha\,\epsilon^{\mu\nu}\zeta_\nu + \frac{1}{2}\,\bar{\theta}\theta\,(i\gamma_\mu f^\mu)_\alpha. \quad (3.20)\end{aligned}$$

The currents $i(\gamma_\mu J^\mu)_\alpha, \zeta^\nu$, etc., thus form a supermultiplet with Θ^μ_μ. Normally the trace identity suffers from an anomaly in theories with ultraviolet divergences. Indeed, $\mathcal{A}_\alpha = 2(\delta S/\delta\Phi)\, D_\alpha \Phi$ turns out nonvanishing at the one-loop level, and this in turn forces a quantum modification of the topological current ζ^ν, as shown in the next section.

One may, using a general real spinor superfield $\omega_\alpha(z)$, cast Eq. (3.19) in the form

$$\int d^4z\,\bar{\omega}_\alpha\Big\{i(\gamma_\mu \mathcal{V}^\mu)_\alpha + 2 D_\alpha W(\Phi)\Big\} = \int d^4z\,\frac{\delta S}{\delta\Phi}\,\delta_\omega \Phi. \quad (3.21)$$

This shows again that the anomalous term $\mathcal{A}_\alpha$ is essentially a Jacobian factor associated with the local field variation

$$\delta_\omega \Phi = 2\bar{\omega}_\alpha D_\alpha \Phi. \quad (3.22)$$

Similarly, the conservation law for $i(\not{x}\mathcal{V}^\mu)_\alpha$ is seen to follow from the field variation

$$\delta\Phi = \delta_\Omega \Phi - 2\,\bar{\omega}_\alpha D_\alpha \Phi, \quad (3.23)$$

where $\delta_\Omega \Phi$ stands for the superfield variation (3.12) with $\Omega_\alpha = -i(\not{x}\omega)_\alpha$. This $\delta\Phi$ correctly involves, e.g., for $\omega_\alpha \to \frac{1}{2}\,\theta_\alpha b$ (with constant b), the field transformation laws under dilatation, $\delta\phi_i = 2b\{x^\mu\partial_\mu\phi_i + (d_i/2)(\partial_\mu x^\mu)\phi_i\}$ with $d_i = (0, 1/2, 1)$ for $\phi_i = (\phi, \psi, F)$. Equation (3.23) isolates from $\delta\Phi$ an anomaly-free variation $\delta_\Omega \Phi$. This shows that the anomalous term $\mathcal{A}_\alpha$, governed by the $\delta_\omega \Phi$ variation, essentially constitutes the superconformal anomaly.

IV. ANOMALIES

In this section we evaluate potentially anomalous equation-of-motion-like terms and discuss their consequences. For operator calculus in superspace it is useful to define the eigenstate of the coordinates $z = (x^\mu, \theta_\alpha)$ by $|z\rangle = |x\rangle|\theta)$, with $\theta_\alpha|\theta') = \theta'_\alpha|\theta')$ and normalization $(\theta'|\theta'') = \frac{1}{2}\,(\bar{\theta}' - \bar{\theta}'')(\theta' - \theta'') \equiv \delta(\theta' - \theta'')$. The covariant derivative D_α is thereby written as an antisymmetric matrix $\langle z|D_\alpha|z'\rangle = D_\alpha \delta(\theta - \theta')\delta^2(x - x') = -\langle z'|D_\alpha|z\rangle$. A direct calculation shows that

$$(\theta|\bar{D}D|\theta') \equiv \bar{D}D\delta(\theta - \theta') = -2\exp[\bar{\theta}\not{p}\theta'], \quad (4.1)$$

where $p_\mu = i\partial_\mu$. Thus, for the θ-diagonal element

$$(\theta| - \frac{1}{2}\bar{D}D|\theta) = 1, \quad (4.2)$$

whereas odd numbers of D's have vanishing diagonal elements $(\theta|D_\alpha|\theta) = (\theta|D_\alpha \bar{D}D|\theta) = 0$, etc., as verified easily. This feature of D's plays an important role in superspace operator calculus.

The potentially anomalous products of our concern have the form

$$\{\Lambda\Phi(z)\}\,\Xi\frac{\delta S}{\delta\Phi(z)}, \quad (4.3)$$

where Λ and Ξ may involve operators D_α and ∂_μ. To evaluate them we use the background-field method [19] and decompose Φ into the classical field $\Phi_{\rm c}$ and the quantum fluctuation χ, $\Phi(z) = \Phi_{\rm c}(z) + \chi(z)$. The quantum fluctuation at the one-loop level is governed by the action

$$\begin{aligned} S_\chi \;&=\; \int d^4z\,\frac{1}{2}\,\chi(z)\,\mathcal{D}\,\chi(z), \quad (4.4)\\ \mathcal{D} \;&=\; -\frac{1}{2}\bar{D}D + W''(\Phi_{\rm c}). \quad (4.5)\end{aligned}$$

The associated χ propagator is given by $i/\mathcal{D}$, which we regularize as

$$\langle\chi(z)\,\chi(z')\rangle^{\rm reg} = \langle z|\frac{i}{\mathcal{D}}\,\Gamma|z'\rangle, \qquad (4.6)$$

$$\Gamma = e^{\tau\mathcal{D}^2}, \qquad (4.7)$$

with $\tau \to 0_+$ in Γ at the very end. This choice $\Gamma = e^{\tau\mathcal{D}^2}$ of the ultraviolet regulator is manifestly supersymmetric and is a natural one [10, 16] that controls quantum fluctuations within the background-field method.

To evaluate Eq. (4.3) at the one-loop level we set $\Lambda\Phi \to \Lambda\chi$ and $\Xi(\delta S/\delta\Phi) \to \Xi(\delta S_\chi/\delta\chi)$, consider as the singular part the expectation value, and substitute the regularized χ propagator (4.6). The result is

$$\langle\{\Lambda\Phi(z)\}\,\Xi\frac{\delta S}{\delta\Phi(z)}\rangle = i\,\langle z|\,\Lambda\Gamma\Xi^T|z\rangle = \pm i\,\langle z|\,\Xi\Gamma\Lambda^T|z\rangle, \qquad (4.8)$$

where the minus sign $-$ applies only when both Λ and Ξ are Grassmann-odd. Here transposition Λ^T is defined by $\langle z|\Lambda^T|z'\rangle \equiv \langle z'|\Lambda|z\rangle$; in particular, $(D_\alpha)^T = -D_\alpha$ and $\Gamma^T = \Gamma$. It is readily seen that the Leibnitz rule applies to the regularized products,

$$(\partial_\mu\Lambda\Phi)\,\Xi\frac{\delta S}{\delta\Phi} + (\Lambda\Phi)\,\partial_\mu\Xi\frac{\delta S}{\delta\Phi} = \partial_\mu\Big[(\Lambda\Phi)\,\Xi\frac{\delta S}{\delta\Phi}\Big]; \qquad (4.9)$$

an analogous formula holds with D_α as well.

From Eq. (4.8) follows a key formula for regularized products,

$$(\Lambda\Phi)\,\Xi\frac{\delta S}{\delta\Phi} = \pm\,(\Xi\Phi)\,\Lambda\frac{\delta S}{\delta\Phi}, \qquad (4.10)$$

where the minus sign applies only when both Λ and Ξ are Grassmann-odd. An immediate consequence of this formula is the conservation of the supercurrent $\partial_\mu\mathcal{V}^\mu_\alpha = 0$; indeed, setting $\Lambda \to D_\alpha\bar{D}_\beta$ and $\Xi \to D_\beta$ in Eq. (4.10) implies the vanishing of the potentially anomalous term in Eq. (3.10), irrespective of the choice of Γ.

Similarly one can show the vanishing of the potentially anomalous products X^μ and r^μ_α [in Eqs. (3.4) and (3.6)] without any direct calculation. To this end, note first the formula

$$\frac{\delta S}{\delta\Phi} = \frac{\delta S}{\delta F} - \bar{\theta}\,\frac{\delta S}{\delta\bar{\psi}} + \frac{1}{2}\,\bar{\theta}\theta\,\frac{\delta S}{\delta\phi}, \qquad (4.11)$$

with which one can relate superfield expressions to component-field expressions. For $X^\mu = i\bar{\psi}\,\gamma^\mu(\delta S/\delta\bar{\psi})$ one may start with the superfield expression $(D_\alpha\Phi)\,D_\beta(\delta S/\delta\Phi)$, which, in view of the formula (4.10), is seen to be antisymmetric in (α,β), or proportional to $(\gamma^0)_{\alpha\beta}$; hence

$$(\bar{D}_\alpha\Phi)\,D_\beta\frac{\delta S}{\delta\Phi} \propto \delta_{\alpha\beta}, \qquad (4.12)$$

which, in component fields, implies that

$$X^\mu = i\bar{\psi}\,\gamma^\mu(\delta S/\delta\bar{\psi}) = 0, \quad \bar{\psi}\,\gamma_5(\delta S/\delta\bar{\psi}) = 0,$$
$$\bar{\psi}\,(\delta S/\delta\bar{\psi}) \neq 0, \text{etc.} \qquad (4.13)$$

For r^μ_α note first the regularized superfield relation following from Eq. (4.10):

$$\frac{1}{2}(\bar{D}D\Phi)\,D_\alpha\frac{\delta S}{\delta\Phi} - (D_\alpha\Phi)\,\frac{1}{2}\bar{D}D\frac{\delta S}{\delta\Phi} = 0, \qquad (4.14)$$

which shows the vanishing of the combination $F\,(\delta S/\delta\bar{\psi}) + \psi\,(\delta S/\delta\phi)$ in r^μ_α. The vanishing of the remaining combination $\psi\,\partial_\lambda(\delta S/\delta F) + (\delta S/\delta\bar{\psi})\,\partial_\lambda\phi$ also follows from an analogous equation with $\frac{1}{2}\bar{D}D$ replaced by ∂_λ in the above. Hence $r^\mu_\alpha = 0$.

On the other hand, the supersymmetric trace identity (3.19) does acquire a quantum modification

$$\mathcal{A}_\alpha \equiv 2\,\frac{\delta S}{\delta\Phi}\,D_\alpha\Phi = iD_\alpha\langle z|\Gamma\,|z\rangle, \qquad (4.15)$$

where the last expression follows from $\mathcal{A}_\alpha = -2i\langle z|\Gamma D_\alpha|z\rangle = 2i\langle z|D_\alpha\Gamma\,|z\rangle = i\langle z|[D_\alpha,\Gamma]\,|z\rangle$. A direct calculation of the relevant heat-kernel is outlined in Appendix B. The result is

$$\mathcal{A}_\alpha = -\frac{1}{2\pi}\,D_\alpha W''(\Phi_{\rm c}). \qquad (4.16)$$

One may now set $\Phi_{\rm c} \to \Phi$ in $\mathcal{A}_\alpha$ to obtain the corresponding operator expression. For the trace identity the net effect of the anomaly is to replace $W(\Phi)$ by $W(\phi) + (1/4\pi)\,W''(\Phi)$ on the right-hand side:

$$i(\gamma_\mu\mathcal{V}^\mu)_\alpha = -2D_\alpha\Big\{W(\Phi) + \frac{1}{4\pi}\,W''(\Phi)\Big\}. \qquad (4.17)$$

For component fields this implies that

$$\begin{aligned}
&i(\gamma_\mu J^\mu)_\alpha = 2F\psi_\alpha = -2\Big(W' + \frac{1}{4\pi}\,W'''\Big)\psi_\alpha,\\
&\Theta^\mu_\mu = F^2 + \frac{i}{2}\,\bar{\psi}\gamma^\mu\partial_\mu\psi\\
&\qquad = -F\Big(W'+\frac{1}{4\pi}\,W'''\Big) + \frac{1}{2}\Big(W''+\frac{1}{4\pi}\,W''''\Big)\bar{\psi}\psi,\\
&\epsilon_{\mu\lambda}\Theta^{\mu\lambda} = \frac{i}{2}\,\bar{\psi}\gamma_5\gamma^\mu\partial_\mu\psi = 0,\\
&\zeta^\mu = -\epsilon^{\mu\nu}F\partial_\nu\phi = \epsilon^{\mu\nu}\partial_\nu\Big(W + \frac{1}{4\pi}\,W''\Big), \text{etc.;}
\end{aligned} \qquad (4.18)$$

these component-field expressions agree with the result of ref. [8].

It is seen from Eq. (4.18) that the role of the auxiliary field F in composite operators is modified at the quantum level: F multiplied by ψ_α or $\partial_\nu\phi$ acts like $-\{W'(\phi) + (1/4\pi)\,W'''(\phi)\}$ rather than $-W'(\phi)$. Note that such modifications take place in the supercurrent (3.3) itself, e.g., in the $F\psi_\beta$ terms of J^μ_α and f^μ_α, and in the $F\partial_\nu\phi$ term of ζ^μ (if one were to eliminate F from these). The naive canonical expressions for ζ^μ and Z in Eqs. (2.17) and (2.20) are not valid any more; ζ^μ in Eq. (4.18) leads to the modified central charge [8]

$$Z = \int dx^1\zeta^0 = \Big[W(\phi) + \frac{1}{4\pi}W''(\phi)\Big]^{x^1=\infty}_{x^1=-\infty}. \qquad (4.19)$$

It is enlightening to look into some other definitions of the superfield supercurrent. Consider a supercurrent with F replaced by $-W'$:

$$\hat{J}^{\mu}_{\alpha} \equiv (\gamma^{\nu}\gamma^{\mu}\psi)_{\alpha}\,\partial_{\nu}\phi + i(\gamma^{\mu}\psi)_{\alpha}\,W'(\phi), \tag{4.20}$$

which differs from J^{μ}_{α} in Eq. (2.13) by $i(\gamma^{\mu}\psi)_{\alpha}\,(\delta S/\delta F)$. This is a supercurrent one would obtain in the on-shell realization of supersymmetry. The superfield generalization of this current is

$$\hat{\mathcal{V}}^{\mu}_{\alpha} = (\partial_{\nu}\Phi)(\gamma^{\nu}\gamma^{\mu})_{\alpha\beta}\,D_{\beta}\Phi + iW'(\Phi)\,(\gamma^{\mu}D)_{\alpha}\Phi, \tag{4.21}$$

which, when expanded in θ as in Eq. (3.3), involves the components

$$\begin{aligned}
\hat{\Theta}^{\mu\lambda} &= \frac{i}{4}\,\bar{\psi}(\gamma^{\mu}\partial^{\lambda} + \gamma^{\lambda}\partial^{\mu})\psi + \partial^{\mu}\phi\partial^{\lambda}\phi \\
&\quad -\frac{1}{2}\,g^{\mu\lambda}\Big\{(\partial\phi)^2 + FW' + \frac{1}{2}\,\bar{\psi}(i\gamma\!\cdot\!\partial - W'')\psi\Big\}, \\
\hat{\zeta}^{\mu} &= (1/2)\,\epsilon^{\mu\nu}(\partial_{\nu}W - F\partial_{\nu}\phi - X_{\nu}), \\
\hat{X}^{\mu} &= F\partial^{\mu}\phi + \partial^{\mu}W + (1/2)\,X^{\mu},\ \text{etc.}
\end{aligned} \tag{4.22}$$

Here we find the symmetric energy-momentum tensor; remember, however, that $\hat{\Theta}^{\mu\lambda}$ and $\Theta^{\mu\lambda}$ are different since $\hat{\Theta}^{\mu}_{\mu} \neq \Theta^{\mu}_{\mu}$ at the quantum level. Also, $\hat{\zeta}^{\mu} = \zeta^{\mu} + (1/8\pi)\epsilon^{\mu\nu}\partial_{\nu}W'' \neq \zeta^{\mu}$ and $\hat{X}^{\mu} = -(1/4\pi)\partial^{\mu}W'' \neq 0$.

The current $\hat{\mathcal{V}}^{\mu}_{\alpha}$ differs from $\mathcal{V}^{\mu}_{\alpha}$ by an anomalous field product, actually by the anomaly $\frac{1}{2}(i\gamma^{\mu}\mathcal{A})_{\alpha}$,

$$\hat{\mathcal{V}}^{\mu}_{\alpha} = \mathcal{V}^{\mu}_{\alpha} + i(\gamma^{\mu}D\Phi)_{\alpha}(\delta S/\delta\Phi). \tag{4.23}$$

As a result, with $\hat{\mathcal{V}}^{\mu}_{\alpha}$, supersymmetry is apparently spoiled while the trace identity looks normal,

$$\partial_{\mu}\hat{\mathcal{V}}^{\mu}_{\alpha} = -(i/4\pi)\,\partial_{\mu}[\gamma^{\mu}D_{\alpha}W''(\Phi)] \neq 0, \tag{4.24}$$

$$(\gamma_{\mu}\hat{\mathcal{V}}^{\mu})_{\alpha} = 2iD_{\alpha}W(\Phi). \tag{4.25}$$

Here we encounter trading of the anomaly between two different conservation laws, a phenomenon observed also in ref. [10] and familiar from nonsupersymmetric theories [18]. Since the trace identity is anomalous in theories with divergences, it is appropriate to recover supersymmetry, which is achieved by passing from $\hat{\mathcal{V}}^{\mu}_{\alpha}$ to $\mathcal{V}^{\mu}_{\alpha}$ through current redefinition. In this sense, $\mathcal{V}^{\mu}_{\alpha}$ is a more natural off-shell definition of the supercurrent.

We have so far studied potentially anomalous products in the one-loop approximation. The basic formula (4.8) with $\Gamma = e^{\tau\mathcal{D}^2}$ itself is an expression valid at the one-loop level. Fortunately it is generalized to higher loops if one notes that the cutoff Γ there derives from the expression

$$\langle z|\Gamma|z'\rangle = -i\mathcal{D}_z\langle\chi(z)\chi(z')\rangle^{\rm reg}; \tag{4.26}$$

that is, one may extract the relevant cutoff Γ from the χ propagator calculated to any desired loop levels using the regularized zero*th*-order χ propagator (4.8).

Actually it is not necessary to study higher-loop corrections in the present case. The central charge anomaly or the supersymmetric anomaly $\mathcal{A}_{\alpha}$ is one-loop exact and there are no higher-loop corrections, as pointed out in ref. [8]. This follows from a dimensional analysis if one notes that the anomalies in local field products come from short distances, as they should: Note that $\mathcal{A}_{\alpha} = 2\,(\delta S/\delta\Phi)\,D_{\alpha}\Phi$ has dimension 3/2 in units of mass while higher-loop corrections, calculated in a fashion outlined above, inevitably supply at least two powers of $W''[\Phi_{\rm c}], W'''[\Phi_{\rm c}], \cdots$, each having dimension one; this implies the one-loop exactness of the anomaly. (No inverse powers of W are allowed as long as the anomaly is short-distance dominated.)

V. SUPERSPACE EFFECTIVE ACTION

In this section we verify the anomaly by a direct calculation of the effective action in superspace. Setting $\Phi(z) \to \Phi_c(z)$ in the action (2.6) yields the classical action $S[\Phi_c]$. The one-loop effective action $\Gamma_1[\Phi_c]$ is a functional of $M = W''(\Phi_c)$, as seen from S_{χ} in Eq. (4.4), and is most efficiently calculated from its derivative $\delta\Gamma_1[\Phi_{\rm c}]/\delta\Phi_{\rm c}(z)$, which is related to the propagator $\langle\chi(z)\,\chi(z')\rangle = i\langle z|(-\frac{1}{2}\,\bar{D}D + M)^{-1}|z'\rangle$,

$$\delta\Gamma_1[\Phi_{\rm c}]/\delta\Phi_{\rm c}(z) = \frac{1}{2}\,M'\langle\chi(z)\,\chi(z)\rangle, \tag{5.1}$$

where $M' \equiv dM/d\Phi_{\rm c} = W'''(\Phi_{\rm c})$. One can evaluate the χ propagator by expanding it in powers of D_{α} acting on M; see Appendix C for details. To $O(D^2)$ the result is

$$\langle\chi(z)\,\chi(z)\rangle = \frac{1}{4\pi}\Big[\ln\frac{\Lambda^2}{M^2} - \frac{\bar{D}DM}{2M^2} + \frac{(\bar{D}_{\alpha}M)(D_{\alpha}M)}{2M^3}\Big], \tag{5.2}$$

where Λ is an ultraviolet cutoff; $\Lambda^2 = e^{-\gamma}/\tau$ if one adopts the regularized propagator (4.6). Substituting this into Eq. (5.1) and integrating with respect to $\Phi_{\rm c}$ then yields the one-loop effective action to $O(D^2)$,

$$\Gamma_1[\Phi_{\rm c}] = \int d^4z\,\frac{1}{8\pi}\Big[M\Big(\ln\frac{\Lambda^2}{M^2} + 2\Big) + \frac{(\bar{D}M)(DM)}{4M^2}\Big]. \tag{5.3}$$

Let us here review briefly how such quantum corrections could relate to the central charge anomaly [1, 8]. As we have seen, supertranslations have no quantum anomaly. This implies that the supermultiplet structure of Q_{α}, P^{μ} and Z in Eq. (3.8), or equivalently the supercharge algebra (2.19), is maintained exactly. For static kink states the superalgebra (2.19) is split into two algebras

$$(Q_1)^2 = P^0 + Z, (Q_2)^2 = P^0 - Z. \tag{5.4}$$

The classical kink solution (2.12), obeying the first-order equation $(\partial/\partial x^1)\phi = W'(\phi)$, is BPS saturated in the sense that it is inert under supertranslations generated by the supercharge Q_2, or equivalently, Q_2 annihilates the kink state [1],

$$Q_2|{\rm kink}\rangle = (P^0 - Z)|{\rm kink}\rangle = 0. \tag{5.5}$$

The supercharge Q_1, on the other hand, acts nontrivially, leading to two degenerate kink states, one bosonic and one fermionic. The BPS saturation (5.5), once established classically, persists [8, 20, 21] at the quantum level (at least in perturbation theory), and the kink mass is related to the central charge $\langle \text{kink}|Z|\text{kink}\rangle$ exactly.

Let us verify the above formal discussion using the effective action $S[\Phi_c]+\Gamma_1[\Phi_c]$. With only the bosonic components $\phi_c(x)$ and $F_c(x)$ of $\Phi_c(z)$ retained, $\Gamma_1[\Phi_c]$ reads

$$\Gamma_1 = \int d^2x\, \frac{1}{8\pi}\Big[F_c M'_\phi \ln\frac{\Lambda^2}{M_\phi^2} + \frac{1}{2}\Big(\frac{M'_\phi}{M_\phi}\Big)^2\{(\partial_\mu\phi_c)^2 + F_c^2\}\Big], \tag{5.6}$$

where $M_\phi \equiv W''(\phi_c)$. Accordingly, the static kink is now governed by the Lagrangian

$$\mathcal{L}_{\rm stat} = -\frac{1}{2}\Big(\sqrt{\alpha}\,(\partial_1\phi_c) \mp \frac{1}{\sqrt{\alpha}}\,W'_{\rm eff}\Big)^2 \mp W'_{\rm eff}\partial_1\phi_c, \tag{5.7}$$

where

$$\alpha(\phi_c) = 1 + \frac{1}{8\pi}\Big(\frac{M'_\phi}{M_\phi}\Big)^2,\;\; W'_{\rm eff}(\phi_c) = W'(\phi_c) + \frac{M'_\phi}{8\pi}\ln\frac{\Lambda^2}{M_\phi^2}. \tag{5.8}$$

This leads to the BPS equation for the kink,

$$\partial_1\phi_c = -F_c = (1/\alpha)\, W'_{\rm eff}(\phi_c), \tag{5.9}$$

with the asymptotic values of ϕ_c at $x^1 = \pm\infty$ now determined by $W'_{\rm eff}(\phi_c) = 0$ and the kink mass related to the central charge $Z_{\rm eff} = \int dx\, W'_{\rm eff}\partial_1\phi_c$,

$$m^{\rm kink} = Z_{\rm eff} = 2W_{\rm eff}(\phi_c)|_{x^1=\infty}, \tag{5.10}$$

where

$$W_{\rm eff}(\phi_c) = W(\phi_c) + \frac{W''}{8\pi}\Big\{\ln\frac{\Lambda^2}{(W'')^2} + 2\Big\}. \tag{5.11}$$

Note here that in the background-field method the superpotential $W(\Phi)$ acquires a one-loop correction of the form,

$$\langle W(\Phi)\rangle = W(\Phi_c) + \frac{1}{2}\, W''(\Phi_c)\langle\chi(z)\chi(z)\rangle + \cdots. \tag{5.12}$$

Using Eq. (5.2) and comparing $\langle W(\Phi)\rangle$ with $W_{\rm eff}(\phi_c)$, one learns that the $(W''/8\pi)\ln[\Lambda^2/(W'')^2]$ term (+ a term $\propto W' \approx 0$) in $W_{\rm eff}$ is essentially a one-loop contribution from $W(\Phi)$ and that the central charge $Z_{\rm eff}$ correctly involves the anomaly term $(1/4\pi)\, W''(\phi)$ of Eq. (4.19).

On the other hand, the same line of argument shows that the equation of motion for Φ_c following from $S[\Phi_c]+\Gamma_1[\Phi_c]$ is neatly summarized by

$$-\frac{1}{2}\,\bar{D}D\Phi_c + \langle W'(\Phi)\rangle = 0. \tag{5.13}$$

This verifies that the operator equation of motion $\delta S/\delta\Phi = 0$ by itself is normal, yielding no anomaly. The effective action thus correctly embodies both the equation of motion and the quantum anomaly.

The ultraviolet cutoff Λ^2 which we have kept so far can be removed by renormalization. Let us consider the Wess-Zumino model with the superpotential (2.11). It is superrenormalizable and only mass renormalization is needed. Let m_r be a finite mass scale and set $m^2 = m_r^2 + \delta m^2$ in the effective action $S[\Phi_c] + \Gamma_1[\Phi_c]$. A convenient choice for the mass counterterm is

$$\delta m^2 = (\lambda^2/\pi)\ln(\Lambda^2/m_r^2), \tag{5.14}$$

the net effect of which is to replace the cutoff Λ^2 by m_r^2 [and m^2 by m_r^2] in $S[\Phi_c] + \Gamma_1[\Phi_c]$ or in $W_{\rm eff}(\phi_c)$ of Eq. (5.11). Minimizing $W_{\rm eff}(\phi_c)$ then gives the kink mass

$$m^{\rm kink} = 2W_{\rm eff}(\phi_c)|_{x^1=\infty} = \frac{m_r^3}{6\lambda^2} - \frac{m_r}{2\pi} \tag{5.15}$$

at $\phi_c(x^1 = \infty) = m_r/(2\lambda)$, in agreement with earlier results [6, 8].

For the sine-Gordon model with $W(\Phi) = mv^2\sin(\Phi/v)$ one may set $m = m_r + \delta m$ and choose

$$\delta m = (m_r/8\pi v^2)\ln(\Lambda^2/m_r^2). \tag{5.16}$$

This leads to $\phi_c(x^1 = \infty) = (\pi/2)\, v$ and the soliton mass

$$m^{\rm sol} = 2W_{\rm eff}(\phi_c)|_{x^1=\infty} = 2m_r v^2 - (m_r/2\pi). \tag{5.17}$$

Finally some remark are in order. In the above we have used the one-loop effective action $\Gamma_1[\Phi_c]$ to $O(D^2)$. Corrections of higher powers of D_α lead to more than two derivatives ∂_1 acting on ϕ_c and hardly affect the asymptotic behavior of ϕ_c for $x^1 \to \pm\infty$, thus leaving the above discussion on BPS saturation intact. It is enlightening to discuss the saturation using superfields. In terms of superfields the BPS saturation (5.5) implies that the bosonic-kink superfield has the form [8]

$$\Phi_c(z) = \phi_c\Big(x^1 - \frac{1}{2}\,\bar{\theta}\theta\Big). \tag{5.18}$$

Substituting this into the equation of motion (5.13) (and noting that $D_1\Phi_c = 0$ and $\bar{D}D\Phi_c = 2\partial_1\phi_c$) yields precisely the BPS equation (5.9); here, unlike the component-field equations of motion, the superfield equation of motion has directly turned into the BPS equation. This offers direct verification that the BPS saturation continues beyond the classical level.

VI. SUMMARY AND DISCUSSION

In the present paper we have studied, within a manifestly supersymmetric setting of the superfield formalism, anomalies and quantum corrections to solitons in two-dimensional theories with $N = 1$ supersymmetry. Extensive use is made of the superfield supercurrent to study the structure of supersymmetry and related superconformal symmetry in the presence of solitons, and

to make explicit the supermultiplet structures of various symmetry currents and their associated anomalies.

Possible anomalies in the conservation laws of the supercurrent and associated currents are calculated from potentially anomalous products of the form (fields)×(equations of motion). Interestingly, such potentially anomalous products appear not only in the current conservation laws but also in the superfield supercurrent itself. This does not imply that the current itself is anomalous.

It is perhaps worthwhile to make this point clear and to summarize how the central charge anomaly arises. In the algebraic (i.e., off-shell) realization of supersymmetry one inevitably handles the auxiliary field F, which in the on-shell realization is eliminated in favor of the basic bosonic field ϕ. The superfield $\mathcal{V}^{\mu}_{\alpha}$ in Eq. (3.1) is an algebraically natural definition of the supercurrent (because it is conserved, as we have seen). In passing to an on-shell expression for the supercurrent one has to replace F by $-W'(\phi) + \delta S/\delta F$. The field products involving F thereby reveal portions that lead to possible deviations from classical expressions. Indeed, the relevant superfield product is

$$-(\bar{D}D\Phi)\, D_{\alpha}\Phi = -D_{\alpha}W(\Phi) + \mathcal{A}_{\alpha} \tag{6.1}$$

in Eq. (3.19), owing to which the product $F\psi_{\beta}$ in J^{μ}_{α} and f^{μ}_{α} has turned into $-\{W' + (1/4\pi)W''\}\psi_{\beta}$, and the product $F\partial_{\mu}\phi$ in ζ^{μ} into $-\{W' + (1/4\pi)W''\}\partial_{\mu}\phi$. The auxiliary field F thus changes its role in each composite operator [22]. This is one way to understand the emergence of anomalies. Actually, since the relevant portion of such changes in the role played by F is associated with superconformal transformations, as we have seen in Sec. III, it may be said that the central charge anomaly is part of the superconformal anomaly. It is important to note here that the supercurrent, untouched in the off-shell realization, is only apparently modified in passing to the on-shell realization of supersymmetry; in this way the supercurrent $\mathcal{V}^{\mu}_{\alpha}$ is capable of accommodating the superconformal anomaly without spoiling supersymmetry.

If, on the other hand, one avoids the use of F, thus necessarily relying on the on-shell realization of supersymmetry, the superfield supercurrent [e.g., $\hat{\mathcal{V}}^{\mu}_{\alpha}$ in Eq. (4.21)] is no longer able to admit the effect of the anomaly and supersymmetry gets apparently spoiled; then redefinition of currents is needed to recover supersymmetry.

The combined use of the background-field method and Fujikawa's path-integral formulation of anomalies is crucial in our discussion of supersymmetric anomalies; remember in this respect that the way the anomalies arise depends critically on the regularization method one uses. The superspace background-field method provides a neat means for supersymmetric calculations [with a very natural regularization prescription (4.6)] and is particularly suited for the discussion of solitons and other topological excitations. As we have seen in Sec. V, it would be advantageous, especially when direct calculations of the soliton mass and/or central charge involve some subtleties, to first study the effective action which reveals how the soliton equation and the associated boundary condition, as well as the central charge, are modified at the quantum level.

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APPENDIX A: SOME USEFUL FORMULAS

In this appendix we summarize some formulas involving spinor derivatives D_{α}. Let us first write, using the algebra $\{\bar{D}_{\alpha}, D_{\beta}\} = 2(\not{p})_{\beta\alpha}$ with $p_{\mu} = i\partial_{\mu}$, products of two D's in the form

$$\begin{aligned} \bar{D}_{\alpha}D_{\beta} &= (\not{p})_{\beta\alpha} + \delta_{\beta\alpha}\,(1/2)\bar{D}D, \\ D_{\alpha}\bar{D}_{\beta} &= (\not{p})_{\alpha\beta} - \delta_{\alpha\beta}\,(1/2)\bar{D}D. \end{aligned} \tag{A1}$$

A direct calculation then shows that D_{α} anticommute with $\bar{D}D$, $\{D_{\alpha}, \bar{D}D\} = 0$. This is combined with an obvious relation $[D_{\alpha}, \{\bar{D}_{\alpha}, D_{\beta}\}] = 0$ to yield

$$\bar{D}_{\alpha}D_{\beta}D_{\alpha} = 0. \tag{A2}$$

Some further formulas useful in operator calculus are

$$\begin{aligned} &(\bar{D}D)^2 = 4\,p^2,\ \ \bar{D}\gamma^{\mu}D = 2\,p^{\mu}, \\ &2(\not{p}D)_{\alpha} = D_{\alpha}\bar{D}D = -\bar{D}DD_{\alpha}. \end{aligned} \tag{A3}$$

APPENDIX B: HEAT-KERNEL IN SUPERSPACE

In this appendix we outline the calculation of the heat-kernel $\Gamma = e^{\tau\mathcal{D}^2}$ with $\mathcal{D} = -\frac{1}{2}\bar{D}D + W''(\Phi_{\rm c})$, relevant to Eq. (4.15). Let us substitute $\mathcal{D}^2 = p^2 - \frac{1}{2}\{\bar{D}D, W''\} + (W'')^2$ into Γ, and expand it in powers of W''. The first-order correction reads

$$\Gamma^{(1)} = -\frac{1}{2}\int_0^{\tau} ds\, e^{(\tau-s)p^2}\{\bar{D}D, W''\}\, e^{sp^2}, \tag{B1}$$

where $W'' = W''[\Phi_{\rm c}]$. On taking the θ-diagonal element, one finds $(\theta| -\frac{1}{2}\{\bar{D}D, W''\}|\theta) = 2W''$. Hence

$$\begin{aligned} \langle z|\Gamma^{(1)}|z\rangle &= 2\langle x|e^{\tau p^2}\int_0^{\tau} ds\, e^{-sp^2}W''e^{sp^2}|x\rangle \\ &= 2\tau\langle x|e^{\tau p^2}W''|x\rangle + \cdots \\ &= \frac{i}{2\pi}\, W''(\Phi_{\rm c}) \quad\ (\tau \to 0_+), \end{aligned} \tag{B2}$$

where we have used

$$\langle x|e^{\tau p^2}|x\rangle = \int \frac{d^2p}{(2\pi)^2}\, e^{\tau p^2} = \frac{i}{4\pi\tau}. \tag{B3}$$

It is readily seen on dimensional grounds that the desired $\langle z|\Gamma|z\rangle$ is given by this first-order result (B2) alone in the $\tau \to 0_+$ limit.

Finally we quote some other examples of anomalous products (in operator form):

$$\begin{aligned}(\bar{D}_\alpha\Phi)D_\beta\frac{\delta S}{\delta\Phi} &= \frac{1}{4\pi}\,(W'')^2\,\delta_{\alpha\beta}, && \text{(B4)}\\ (\bar{D}D\Phi)\frac{\delta S}{\delta\Phi} &= -\frac{1}{4\pi}\left[\bar{D}DW'' + 2(W'')^2\right], && \text{(B5)}\\ G(\Phi)\,\frac{\delta S}{\delta\Phi} &= -\frac{1}{2\pi}\,G'(\Phi)W''(\Phi), && \text{(B6)}\end{aligned}$$

where $G(\Phi)$ is a polynomial of Φ. These imply that $\bar{\psi}\,(\delta S/\delta\bar{\psi}) \neq 0$, $F(\delta S/\delta F) = \phi\,(\delta S/\delta\phi) \neq 0$, etc.

APPENDIX C: PROPAGATOR

In this appendix we outline the calculation of the propagator $\langle\chi(z)\chi(z)\rangle = i\langle z|(-\frac{1}{2}\bar{D}D + M)^{-1}|z\rangle$ with $M = W''(\Phi_c)$ in Eq. (5.2). Let us rewrite the propagator in the form

$$\langle z|\frac{i}{p^2 - (\bar{D}M)D - \kappa^2}(-\frac{1}{2}\bar{D}D - M)|z\rangle \tag{C1}$$

with $\kappa^2 = \frac{1}{2}(\bar{D}DM) + M^2$. We then expand it in powers of $(\bar{D}M)D$, bring the derivatives D_α to the very right-hand side, and note formulas such as $(\theta| - \frac{1}{2}\bar{D}D|\theta) = 1$. This leads to expressions in ordinary space-time (x^μ), classified in powers of D_α acting on M. Retaining terms to $O(D^2)$ yields

$$\langle x|\frac{i}{p^2 - \kappa^2}|x\rangle + \langle x|\frac{i}{(p^2-\kappa^2)^3}|x\rangle M(\bar{D}M)(DM), \tag{C2}$$

which eventually leads to Eq. (5.2).

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